Sizes of Infinity *(a quick math crash course)*

There is an infinite number of possible ***binary strings*** (`0`, `1`, `01010`, `111100`, `11`, ...) and there is also an infinite number of possible ***lists of binary strings*** ({`1`, `11`, `111`, ...}, {`0`, `01`, `001`, ...}, ...).

Our question: *Are they the same infinity?* In 1891, Georg Cantor proved that they weren't!

Binary strings, and countable infinity

We can write *all* the possible binary strings in order, and count them up one by one:

1	""	
2	"0"	
3	"1"	First the strings that are one digit long...
4	"00"	
5	"01"	...then the strings that are two digits long...
6	"10"	
7	"11"	...then three digits...*and so on*
8	"000"	

Each possible binary string can be *uniquely* assigned to each counting number (1, 2, 3, ...). Because of this, we say that the number of binary strings is ***countable infinity***, because we can count off all the elements one by one. Mathematicians sometimes write this as $\aleph_0$.

Lists of binary strings, and uncountable infinity

Can you write out all of the possible *lists of* binary strings in order, and count them? Here's what happens when we try to count them by putting the lists in some order.

Along the top I have written all of the possible binary strings. For each list, I write "Y" if a list contains a certain binary string, or "N" if it doesn't.

	""	"0"	"1"	"00"	"01"	"10"	"11"	"000"	...
List 1	***Y***	Y	Y	Y	Y	N	Y	Y	...
List 2	Y	***N***	N	N	N	N	Y	N	...
List 3	Y	Y	***Y***	N	N	N	Y	Y	...
List 4	N	N	N	***N***	N	N	N	N	...
List 5	Y	Y	Y	Y	***Y***	Y	Y	Y	...
List 6	Y	N	Y	N	Y	***N***	Y	N	...
List 7	N	N	Y	Y	N	N	***Y***	Y	...
...	...	...	...	...	...	...	...	...	...

Now we do something devious. We look at the Ys/Ns in the *diagonal* of this gigantic counting that I have made. Then, we make a new list, which *reverses* all of the Ys/Ns in that diagonal:

TrollList	***N***	***Y***	***N***	***Y***	***N***	***Y***	***N***	...	...

TrollList *isn't* in our counting! It can't be, because it is by definition different from every single list we have counted. We cannot count the number of lists, it's impossible!

This is called ***Cantor's diagonalization argument***, published in 1891 by Georg Cantor.

It also doesn't matter what ordering we make in the first place – when we make TrollList, it will *always* be an example of a list that we have failed to count.

The number of lists of binary strings is an ***uncountable infinity***, because we just cannot count them. Mathematicians sometimes write this number as $2^{\aleph_0}$.

Test Yourself What are the sizes of each of these infinities? (circle one)

The number of binary strings	Countable ∞ / Uncountable ∞
The number of lists of binary strings	Countable ∞ / Uncountable ∞
The number of positive integers (1, 2, 3, 4, ...)	Countable ∞ / Uncountable ∞
The number of integers (... -2, -1, 0, 1, 2, ...)	Countable ∞ / Uncountable ∞
The rational numbers (Anything that can be expressed as *an integer ÷ an integer.*) (1.3333, 77.25, 5.0, -37.1, etc.)	Countable ∞ / Uncountable ∞
The real numbers between 0 and 1 (0.1248..., 0.99212..., 0.05127..., etc.) (This includes the ones that are infinite without patterns, such as π/4.)	Countable ∞ / Uncountable ∞

Fun FAQs

→ *Is there a size of infinity larger than* $2^{\aleph_0}$*?*

Yes, there is!

Think about the number of lists of lists of binary strings. Using a similar argument to the we gave above, we can demonstrate that this number must be greater than the number of lists of binary strings. We call this number $2^{2^{\aleph_0}}$.

And we can keep going like this... there are an infinite number of sizes of infinity!

→ *Is there a size of infinity between* $\aleph_0$ *and* $2^{\aleph_0}$*?*

Well, it's complicated. In 1878, Georg Cantor thought about this question, and guessed that there were *not* any sizes of infinity between $\aleph_0$ and $2^{\aleph_0}$. This hypothesis was named *the Continuum Hypothesis.*

1. In 1940, Kurt Gödel proved that it was impossible to disprove the Continuum Hypothesis using standard set theory.

Okaaay... so... hey, mathematicians, can you do anything about *proving* it?

2. In 1963, Paul Cohen proved that it was also impossible to prove the Continuum Hypothesis using standard set theory.

Weird, right?

The Sea of Uncomputability.

Test Yourself What are the sizes of each of these infinities? (circle one)

The number of binary strings	Countable ∞ / Uncountable ∞
The number of possible Turing machines	Countable ∞ / Uncountable ∞
The number of lists of binary strings	Countable ∞ / Uncountable ∞
The number of possible languages	Countable ∞ / Uncountable ∞

Uncomputability. Use the above to think about these questions.

How many *total* possible problems are there? (Why?)

How many *solvable* problems are there? (Why?)

Can we build a computer to solve every possible problem? (Why not?)

We call problems that can't be solved **uncomputable**, or **undecidable**.

The Halting Problem is just a single drop in the vast, infinite sea of uncomputable problems: